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TECHNICAL NOTES

Heat and mass transfer to a turbulent liquid film

A. FAGHRI
Department of Mechanical Systems Engineering, Wright State University, Dayton, OH 45435, U.S.A.

and

R. A. SEBAN
Department of Mechanical Engineering, University of California, Berkeley, CA 94720, U.S.A.

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INTRODUCTION

HEAT AND mass transfer to a fully developed turbulent film flow of a binary solution down a wall has been considered by Grossman and Heath [1] in terms of a numerical solution of the energy and species equations in an approximate, constant property form. The initial temperature and concentrations are uniform, and at the liquid–gas phase interface one component of the solution is in equilibrium with its vapor in the saturation state. At the wall the temperature is invariant at the initial temperature or the heat flux is zero (adiabatic wall). Among the various results given in ref. [1], the local Sherwood and Nusselt numbers for the interface are shown graphically. There were errors in some of these results and the asymptotic values of the Sherwood and Nusselt numbers for larger distances from the origin are given in ref. [2] which is the errata to ref. [1].

The dependence of the local Sherwood and Nusselt numbers for the interface with distance, as shown in ref. [1], is rather typical of that for any fully developed internal flow with constant wall temperature in which the transfer number diminishes with distance at first and attains an asymptotic value for larger distances. This exists also for the combined heat and mass transfer case as considered here despite the fact that the interface temperature and concentrations vary with distance even in the region in which an asymptotic value of the transfer number exists. Because of the relatively small variation of the transfer coefficients it is appropriate to consider them to be constant, under which circumstances an analytical result can be obtained. The use of the asymptotic values of the Nusselt and Sherwood numbers at the interface as given by Grossman and Heath [2], together with an estimate of the Nusselt number at the wall, gives results for a Reynolds number of 10^4 that compare quite well with those given for a selection of parameters in ref. [1]. For the other Reynolds number of 10^5 for which ref. [1] gives solutions for only one set of parameters the comparison with the analytical solution is less favorable.

This presentation outlines the analysis and considers the specifications of the required Nusselt and Sherwood numbers and also gives the comparison of the analytical solution with the numerical results of ref. [1].

ANALYSIS

This analysis uses nomenclature that corresponds in general to that of ref. [1]. In ref. [1], constant transport properties and constant density were assumed for the solution and the change in the mass flow rate due to mass transfer into the film was neglected

$$v\frac{\partial \gamma}{\partial \zeta}=\frac{\partial}{\partial \eta}\left[\left(\frac{1}{Sc}+\frac{\varepsilon_{\mathrm{D}}}{\nu}\right)\frac{\partial \gamma}{\partial \eta}\right] \tag{1}$$

$$v\frac{\partial \theta}{\partial \zeta}=\frac{\partial}{\partial \eta}\left[\left(\frac{1}{Pr}+\frac{\varepsilon_{\mathrm{H}}}{\nu}\right)\frac{\partial \theta}{\partial \eta}\right] \tag{2}$$

where

$$v=\frac{u}{\bar{u}},\quad \eta=\frac{y}{\Delta},\quad \zeta=\frac{4\mu}{4\Gamma}\frac{x}{\Delta}=\frac{4}{Re}\frac{x}{\Delta}.$$

Integration of equations (1) and (2) from the wall, $\eta = 0$, to the interface, $\eta = 1$, gives the following relations, where $\bar{\gamma}$ is the mixed mean concentration, and similarly $\bar{\theta}$ is the mixed mean temperature:

$$\frac{\mathrm{d}\bar{\gamma}}{\mathrm{d}\zeta}=\frac{1}{Sc}\left[\frac{h_{\mathrm{D}}\Delta}{D}\right]_{1}(\gamma_1-\bar{\gamma})\equiv S(\gamma_1-\bar{\gamma}) \tag{3}$$

$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\zeta}=N_1(\theta_1-\bar{\theta})-N_0(\bar{\theta}-\theta_{\mathrm{w}})$$

$$\left[\frac{h_{\mathrm{d}}\Delta}{D}\right]_{1}=\frac{\left.\frac{\partial \gamma}{\partial \eta}\right|_{1}}{(\gamma_1-\bar{\gamma})},\quad \left[\frac{h\Delta}{k}\right]_{1}=\frac{\left.\frac{\partial \theta}{\partial \eta}\right|_{1}}{(\theta_1-\bar{\theta})}=N_1\,Pr \tag{4a}$$

$$\left[\frac{h\Delta}{k}\right]_{0}=\frac{\left.\frac{\partial \theta}{\partial \eta}\right|_{0}}{(\bar{\theta}-\theta_{\mathrm{w}})}=N_0\,Pr.$$

If the heat flux, q_{w}, into the fluid at the wall is prescribed instead of the wall temperature, T_{w}, then the energy equation is

$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\zeta}=N_1(\theta_1-\bar{\theta})+Q\quad\text{where}\quad Q\equiv\frac{q_{\mathrm{w}}\Delta}{k(T_{\mathrm{e}}-T_0)}\left(\frac{\alpha}{\nu}\right). \tag{4b}$$

At the interface, $\eta = 1$, the energy liberated by the inward flow of the solvent equals the inward flow of heat

$$\gamma_1-\bar{\gamma}=\frac{N_1}{SL}(\theta_1-\bar{\theta}),\quad L\equiv\frac{\alpha}{D}\lambda\quad\text{and}$$

$$\lambda\equiv\left[(h_{\mathrm{g}}-H_1)\frac{D}{k}\frac{C_{\mathrm{e}}-C_0}{T_{\mathrm{e}}-T_0}\right]. \tag{5}$$

The difference between the interface (equilibrium) temperature of the solution and the saturation temperature of the pure solute depends on the concentration at the surface. If this difference is linear in concentration, $T_1-T_{\mathrm{sat}}=Rx_1$, then

$$\theta_1+\gamma_1 = 1. \tag{6}$$

A combination of equations (5) and (6) relates γ_1 to $\bar{\gamma}$ and $\bar{\theta}$ as

$$\gamma_1 = \frac{\bar{\gamma}}{a}+b(1-\bar{\theta}), \quad \text{where} \quad a = 1+\frac{N_1}{SL}; \quad b = \left(1-\frac{1}{a}\right). \tag{7}$$

Using equation (7), and $D^* \equiv d/d\zeta$, gives equations (3) and (4a) in the form

$$(D^*+Sb)\bar{\gamma}+Sb\,\bar{\theta} = Sb \tag{8}$$

$$(D^*+N_1+N_0-bN_0)\bar{\theta}+\frac{N_1}{a}\bar{\gamma} = N_1(1-b)+N_0\theta_w \tag{9a}$$

or, for equation (4b)

$$[D^*+N_1(1-b)]\bar{\theta}+\frac{N_1}{a}\bar{\gamma} = N_1(1-b)+Q. \tag{9b}$$

The combination of equations (8) and (9a) produces a second-order ordinary differential equation in which the coefficients are constant if the transfer coefficients are constant. With this specification, the solution for constant wall temperature is

$$\bar{\gamma} = (1-\theta_w)+c_1 e^{r_1\zeta}+c_2 e^{r_2\zeta} \tag{10}$$

$$\bar{\theta} = \theta_w-c_1\left[\frac{r_1}{Sb}+1\right]e^{r_1\zeta}-c_2\left[\frac{r_2}{Sb}+1\right]e^{r_2\zeta} \tag{11}$$

where

$$A = Sb, \quad A+B = Sb+\frac{N_1}{a}+N_0, \quad AB-\frac{Sb\,N_1}{a} = Sb\,N_0$$

$$r_{1,2} = \frac{A+B}{2}\left[-1\pm\sqrt{\left(1-\frac{4Sb\,N_0}{(A+B)^2}\right)}\right]$$

$$c_1 = \frac{Sb+r_2(1-\theta_w)}{(r_1-r_2)}$$

$$c_2 = -\left(\frac{Sb+r_1(1-\theta_w)}{(r_1-r_2)}\right).$$

Then equation (7) gives the interface concentration, γ_1 (which is $1-\theta_1$), as

$$\gamma_1 = (1-\theta_w)+c_1\left[1+\frac{r_1}{S}\right]e^{r_1\zeta}+c_2\left[1+\frac{r_2}{S}\right]e^{r_2\zeta}. \tag{12}$$

For the second case considered, that of constant heat flux, q_w, at the wall, $\bar{\gamma}$, $\bar{\theta}$ and γ, are given by the following solutions of equations (8) and (9b):

$$\bar{\gamma} = \frac{Sb}{A+B}\left[-Q\zeta+\left(1+\frac{Q}{A+B}\right)(1-\exp(-(A+B)\zeta))\right] \tag{13}$$

$$(1-\bar{\theta}) = \frac{1}{1+L}\left[-\frac{Q}{Sb}-Q\zeta+\left(1+\frac{Q}{A+B}\right)L\,e^{-(A+B)\zeta}\right] \tag{14}$$

$$\gamma_1 = \frac{1}{1+L}\left[1-\left(1+\frac{A+B}{S}\right)e^{-(A+B)\zeta}\right]$$

(for adiabatic wall only) (15)

where for this case $A = Sb$, and as before $B = N_1(1-b)$

$$A+B = Sb+\frac{N_1}{a} = \frac{SN_1(1+L)}{SL+N_1}, \quad \frac{Sb}{A+B} = \frac{1}{1+L}.$$

The use of these results to predict $\bar{\gamma}$, $\bar{\theta}$ and γ_1 requires values of λ, the Nusselt number and the Sherwood number at the interface and the Nusselt number at the wall, or the value of Q if a constant flux is specified at the wall. Actually, the Sherwood and Nusselt numbers depend upon the length in the initial regions of concentration and temperature profile development and the average values would exceed the asymptotic values that exist for large lengths. With turbulent flow the development region is relatively short and the asymptotic values of the Sherwood and Nusselt numbers are expected to produce adequate predictions except for very small distances from the initial position. There follows an indication of how these values are to be obtained at the interface and the wall for given solution properties and film Reynolds numbers.

TRANSFER COEFFICIENTS

For the Sherwood number at the interface, a number of relations exist. One is that of Won and Mills [3] for gas as the solute and various liquids as the solvent, flowing down a vertical wall. Schmidt numbers ranged from 83 to 2700 and for the turbulent Reynolds numbers from 2000 to 8000. For turbulent flow the relation given by Won and Mills [3] is as follows:

$$\frac{h_D\Delta}{D} = 6.97\times10^{-9}\left(\frac{4\Gamma}{\mu}\right)^m\left(\frac{\nu}{D}\right)^\beta(\Delta^+)^{2/3}(Cb)^{-2} \tag{16}$$

where

$$Cb = \nu\left(\frac{\rho^3 g}{\sigma^3}\right)^{1/4}; \quad m = 3.49(Cb)^{0.27}$$

and

$$\beta = 0.64-2.43\sigma$$

with the surface tension, σ, in N m^{-1}.

For the Nusselt number at the interface ($h\Delta/k$), equation (16) evaluated with the Schmidt number taken as the Prandtl number suffices if the Prandtl number is sufficiently large. For lower Prandtl numbers there is no available information for the boundary conditions of interest and estimates involve, among other effects, a definition of the nature of the eddy diffusivity in the region of the interface. Such definitions have often been made by assuming that in this region, for mass transfer: $\varepsilon_D/\nu = E(y'/\Delta)^n$, where y' is the distance from the interface into the film.

With this, the mass flux is

$$\frac{j}{D} = \left[1+\frac{\nu}{D}E\left(\frac{y'}{\Delta}\right)^n\right]\frac{\partial(x_1-x)}{\partial(y')}.$$

If it is assumed that this region is thin enough so that the flux j is invariant with y', then an integration which neglects any dependence of concentration distribution with distance ζ gives

$$\frac{D(x_1-x)}{j\Delta} = \int_0^{y'/\Delta}\frac{d\left(\frac{y'}{\Delta}\right)}{1+\frac{\nu}{D}E\left(\frac{y'}{\Delta}\right)^n} = \frac{n}{\left(\frac{\nu}{D}E\right)^{1/n}}\int_0^{\xi}\frac{(\xi^{1/n}-1)\,d\xi}{(1+\xi)} \tag{17a}$$

where

$$\xi = \frac{\nu}{D}E\left(\frac{y'}{\Delta}\right)^n \quad \text{or} \quad \left(\frac{\nu}{D}\frac{\varepsilon_D}{\nu}\right).$$

The integral in equation (17a) is an incomplete Beta function but for $n = 2$ it gives a simple algebraic value to make equation (17a)

$$\frac{j\Delta}{(x_1-x)D} = \frac{\left(\frac{\nu}{D}E\right)^{1/2}}{\tan^{-1}\sqrt{\left(\frac{\nu}{D}\frac{\varepsilon_D}{\nu}\right)}}. \tag{17b}$$

For the value of the Schmidt number, the value of the arc tangent is $\pi/2$ even for $\varepsilon_D/\nu \rightarrow 1$. It is assumed that at this distance, y', the concentration is the mixed mean value and that equation (17b) gives the Sherwood numbers. Then a combination of equation (17b) with a correlation for the Sherwood number specifies the factor E. If equation (16) is used for the correlation then besides a Reynolds number dependence, E depends also on the Schmidt number and the properties of the liquid of the film. In ref. [1] it is implied that E is taken as

$$E = 2\times10^{-6}(Re)^{1.425}(\Delta^+)^{4/3}. \tag{18}$$

The use of E in equation (17a) will produce Sherwood numbers like those from equation (16), presumably for water at about 25°C.

For small Schmidt or Prandtl numbers it is assumed that the domain in which equation (17b) applies ends where the eddy diffusivity obtains the relatively constant value that exists in the central region of the turbulent flow and that this domain, together with the sublayer region near the wall, are together small enough so that the central region of the turbulent flow is all at the mixed mean concentration (temperature). If the value of the diffusivity in the core region is defined, then equation (17b) with equation (18) gives the Sherwood or Prandtl number. Szablewski [4] indicates a momentum diffusivity for this central turbulent region of film flow in the following form:

$$\frac{\varepsilon_m}{\nu} = 0.057\frac{\Delta\sqrt{\left(\frac{\tau_0}{\rho}\right)}}{\nu} = 0.057\Delta^+. \tag{19}$$

Taking $\varepsilon_D = \varepsilon_m$, equation (17b) gives the Nusselt number as

$$\left(\frac{h\Delta}{K}\right)_1 = \frac{1.414\times10^{-3}\left(\frac{\nu}{\alpha}\right)^{1/2}(Re)^{0.712}(\Delta^+)^{2/3}}{\tan^{-1}\sqrt{\left(0.057\frac{\nu}{\alpha}\Delta^+\right)}}. \tag{20}$$

A Nusselt number is also needed for the wall when the wall temperature is constant or for the determination of the wall temperature when the heat flux at the wall is constant. In the spirit of what was done above an appropriate sublayer model would suffice to define a transfer coefficient but rather there is used for the wall the Nusselt number given by the Dittus–Boelter correlation, written for parallel plates a distance of 2Δ apart

$$\left(\frac{h\Delta}{k}\right)_0 = \frac{0.023}{4}\left(\frac{4\Gamma}{\mu}\right)^{0.8}\left(\frac{\nu}{\alpha}\right)^{1/3}. \tag{21}$$

The specification of the Nusselt and Sherwood numbers at the interface requires the relation of Δ^+ to the Reynolds number. A variety of such relations is contained in ref. [5] by Seban, with extremes typified by the correlations of Brotz and of Brauer

$$\Delta^+ = 0.0174\left(\frac{4\Gamma}{\mu}\right) \quad \text{Brotz} \tag{22}$$

$$\Delta^+ = 0.095\left(\frac{4\Gamma}{\mu}\right)^{0.8} \quad \text{Brauer.} \tag{23}$$

In Table 1 both equations (22) and (23) are used with equation (20) to predict the Nusselt and Sherwood numbers at the interface for the two Reynolds numbers for which these are given in ref. [2]. Neither ref. [1] nor ref. [2] gave the values of the Nusselt numbers at the wall so that no comparison with equation (21) is possible. The comparisons in Table 1 are good for a Reynolds number of 10^4 particularly since both equations (22) and (23) give values that are nearly the same. This is not true for the Reynolds number of 10^5, for which the higher values of Δ^+ given by equation (22) give Sherwood and Nusselt numbers 1.5 times those from equation (23). The latter results agree best with those from ref. [2].

COMPARISON

The graphical results of ref. [1] give $\bar{y}$, $\bar{\theta}$ and θ_1, which is $(1-y_1)$, as functions of ζ, the more complete portrayal being for the combination of $(\nu/\alpha) = 10$, $(\nu/D) = 2000$, $\lambda = 0.01$, for $(4\Gamma/\mu) = 10^4$. Table 2 gives values obtained from the figures in ref. [1] both for the isothermal wall and the adiabatic wall cases. For all values of the parameters other than those cited above, only the interface values, θ_1, were given in ref. [1]. The subscript G is used to identify values from ref. [1]. For comparison there are the values from the analytical solution, which were evaluated using the asymptotic values of the Sherwood and Nusselt numbers at the interface as given in ref. [2], and contained in Table 1. The wall Nusselt numbers required for the isothermal wall case were obtained from equation (21) and are 20 and 9.1 for Prandtl numbers of 10 and 1, respectively.

In general, the correspondence of the two sets of values is mostly within the accuracy with which the values from ref. [1] can be read from the figures that are given. For the higher Reynolds number of 10^5, only $\bar{y}$ is given in ref. [1] for the isothermal wall and adiabatic wall cases for $\nu/\alpha = 10$, $\nu/D = 2000$, $\lambda = 0.01$. Typical values are given in Table 3. Columns (a) of this table give the values of $\bar{y}$ predicted by the analytical results, using the interface Sherwood and Nusselt numbers from ref. [2], as given in Table 1, with the wall Nusselt number obtained from equation (21) being equal to 124. These results for $\bar{y}$ are much lower than those of the numerical solutions as given in ref. [1]. For this higher

Table 1

Re	Sc	(1) Sh	(2) Sh	(3) Sh	(4) Sh	(5) Sh	(6) Nu_1	(7) Nu_1	(8) Nu_1	Pr
10^4	200	276	280	258	245	222	19.5	25	24	1
	2000	873	885	816	717	650	61.7	68	63	10
10^5	200	4863	6693	4468	5093	3400	344	516	354	1
	2000	15380	21165	14129	14900	9950	1087	1535	1038	10

(1) From ref. [2].
(2) Equation (20) using Brotz for Δ^+.
(3) Equation (20) using Brauer for Δ^+.
(4) Equation (16) with Brotz $T = 25$°C.
(5) Equation (16) with Brauer $T = 25$°C.
(6) From ref. [2].
(7) Using Brotz for Δ^+.
(8) Using Brauer for Δ^+.

Table 2

					Isothermal						Adiabatic					
Re	*Pr*	*Sc*	λ	ζ	$\bar{y}$	$\bar{y}_G$	$\bar{\theta}$	$\bar{\theta}_G$	θ_I	θ_{IG}	$\bar{y}$	$\bar{y}_G$	$\bar{\theta}$	$\bar{\theta}_G$	θ_I	θ_{IG}
10^4	10	2000	0.01	0.2	0.07	0.07	0.12	0.12	0.20	0.22	0.07	0.06	0.13	0.12	0.23	0.25
				1	0.26	0.26	0.22	0.16	0.28	0.26	0.23	0.22	0.45	0.45	0.49	0.50
				2	0.43	0.45	0.18	0.13	0.23	0.20	0.30	0.28	0.60	0.60	0.61	0.61
				4	0.66	0.67	0.11	0.09	0.14	0.12	0.33	0.33	0.66	0.66	0.67	0.67
					Isothermal			Adiabatic				Isothermal			Adiabatic	
			$\lambda = 0.1$	ζ	θ_I	θ_{IG}		θ_I	θ_{IG}	$\lambda = 0.001$	ζ	θ_I	θ_{IG}		θ_I	θ_{IG}
				0.2	0.75	0.74		0.79	0.78		0.2	0.03	0.03		0.03	0.03
				1	0.79	0.76		0.94	0.94		1	0.03	0.03		0.08	0.08
				2	0.74	0.72		0.95	0.95		2	0.02	0.02		0.12	0.12
				4	0.66	0.70		0.95	0.95		4				0.15	0.15
10^4	10	200	0.01	0.2					0.07	0.07					0.07	0.08
				0.4					0.08	0.08					0.10	0.10
				1.0					0.06	0.06					0.14	0.14
	1	2000	0.01	0.02					0.38	0.38					0.38	0.38
				0.04					0.43	0.45					0.43	0.45
				0.10					0.52	0.56					0.62	0.62
				0.40					0.55	0.59					0.90	0.90
				1.0					0.50	0.56					0.95	0.95
				4					0.29	0.34					0.95	0.95
				10					0.10	0.12						

Table 3

						Isothermal			Adiabatic		
						(a)	(b)	(c)		(a)	(b)
10^5	10	2000	0.01	ζ	$\bar{y}_G$	$\bar{y}$	$\bar{y}$	$\bar{y}$	$\bar{y}_G$	$\bar{y}$	$\bar{y}$
				0.01	0.12	0.06	0.12	0.01	0.12	0.06	0.12
				0.02	0.20	0.11	0.18	0.21	0.20	0.11	0.20
				0.04	0.28	0.20	0.27	0.29	0.27	0.18	0.28
				0.10	0.47	0.35	0.40	0.48	0.33	0.33	0.33
				0.40	0.83	0.73	0.76	0.88			

(a) *Nu*, *Sh* from ref. [2] (Table 1) and $Nu_0 = 124$.
(b) *Nu*, *Sh* twice [2] (Table 1) and $Nu_0 = 124$.
(c) *Nu*, *Sh* twice [2] (Table 1) and $Nu_0 = 248$.

Reynolds number, the assumption that the Nusselt and Sherwood numbers being constant at their asymptotic values should be even more appropriate than at the lower Reynolds number of 10^4, for which a good correspondence was obtained. Therefore, for the higher Reynolds number, either the results of ref. [1] for $\bar{y}$ or those of ref. [2] for the transfer numbers appear to be in some question. The values of the transfer numbers at the interface as obtained from equation (20) depend substantially on whether equation (21) or equation (22) is used to speficy Δ^+. A further check on this value as implied for the results in ref. [1] is not possible since, though Δ^+ was calculated implicitly in ref. [1], its value is not given there. Nothing decisive can be said about these uncertainties but as an indication Table 3 contains in column (b) the values for $\bar{y}$ from the analytical solution for the Nusselt and Sherwood numbers at the interface twice those used for columns (a). This improves the comparison to the results of ref. [1]. It is shown by column (c) that also doubling the value of the Nusselt number for the wall also produces a slight increase in the value of $\bar{y}$. The difficulty associated with these observations is that, as indicated by Table 1, the Nusselt and Sherwood numbers at the interface that are required are much higher than would be predicted by the methods suggested for that purpose.

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